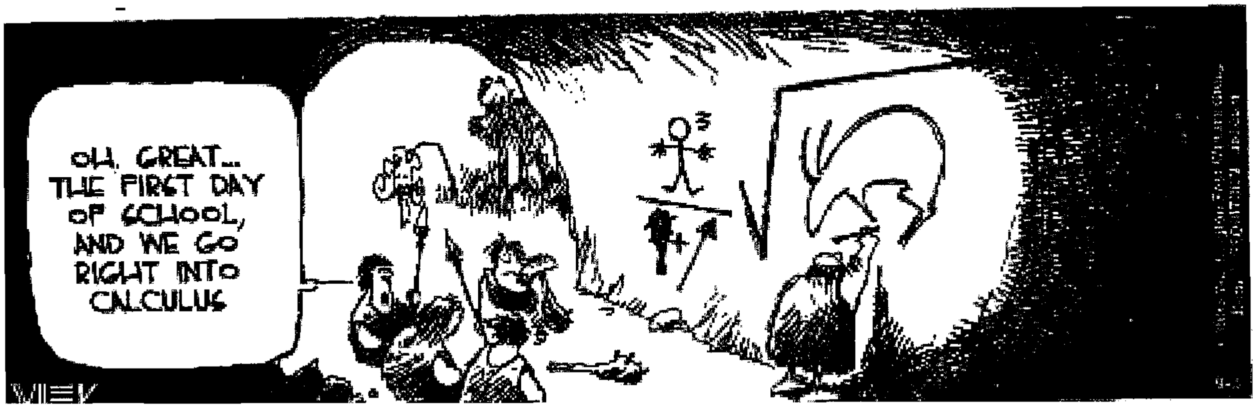


Calculus for AP Physics Students

We begin our journey of learning advanced physics by increasing our ability to solve more complex math problems. These problems are similar to our earlier simple physics problems in that they involve looking at motion and how it changes and is influenced. However, now we need to look at rates of changes that are not constant. We have only studied constant velocity, acceleration, forces, and other rates and the problems were reasonably easy to solve like falling objects, etc. The formulas for these constant rates of change were also easy to calculate and graph. Now we must learn how to deal with changing "rates of change" and their more complex formulas and solutions. In order to do this, we must learn more complex processes of solving for variable rates of change. We call this process calculus, a math process developed by Isaac Newton and Gottfried Leibniz in the late 1600's. These two guys built upon other mathematical foundations and developed something that changed the world of math. In order for us to solve even the simplest variable rate change problems in physics, we need to have a rudimentary working knowledge of calculus. We don't need to know the history or derivation of the process, just how to use it. I like the analogy of building a new shower for your house. We don't have to know how to put the pipes and parts together, we just need to know how to turn on the faucet. You will learn all the background and derivations in your calculus class.

***Special Note: Please do all the practice problems and make sure you understand the process and the solutions. I will test you on them constantly in the first few weeks and expect you to be on top of things. Also, this is not a place to just copy someone else's work or just glance over the material. It would be like watching someone else lift weights hoping you would get stronger.**

A characteristic (maybe an attribute) of mathematicians is an urge to explain in detail all their work. For us, there is just too much material to go through, so we will skip the history, philosophy, extended discussions, proofs, and derivations (almost). It is important, however, that you learn the details from your calculus classes.

We will focus on procedures and drills for solving physics problems.

Going back to regular physics, we will refer to a situation where we are tossing a ball straight up in the air and letting it fall back to our hand. Remember that the ball starts off with an initial velocity and then slows down until reaching its apex and then begins to increase its velocity again while falling. So.....

We will look at ball's changing velocity and the concept of the *derivative.*
Say the ball travels in *meters* and *time* is in seconds

The distance the ball travels is a function,

$f(t) = 5t^2$ f(t) is <u>distance</u> as a function of time (we usually say x(t))

after .1 sec ball falls $f(0.1) = 5\,(0.1)^2 = 0.05$ m
after 1.0sec ball falls $f(1) = 5\,(1)^2 = 5$ meters
Hey, we knew that!!!!

So we realize that the ball is speeding up or slowing down but always changing its velocity and covering more or less distance with respect to time.

Now we need to find a formula that describes the ball's velocity at every instant (or given instant).

We can do this by using CALCULUS

Now the tricky thing about computing velocity at an instant in time is that the distance traveled and the elapsed time are both zero! That is distance traveled = 0 and time change = 0

so $\frac{d}{t} = \frac{0}{0}$ **is not defined and does not work!**

But we can use the idea of *average velocity* where

$$\frac{\text{Total distance}}{\text{Total time}} = \text{average velocity}$$

But average means changing!

So, we must compute the average velocity over smaller and smaller time periods, Δt, until we get really, really, really close to an instantaneous velocity

say $\Delta t = 0.001$ **(milliseconds) or smaller**

So let's calculate the distance traveled by the ball between time t and $(t + \Delta t)$ where $(t + \Delta t)$ is a very small period of time.

O $f(t)$	**Now use the average velocity idea where**
↓	**change dist** $= f(t + \Delta t) - f(t)$
↓	
↓	**and** $f(t) = 5t^2$ **so** $= 5(t + \Delta t)^2 - 5t^2$
↓	
↓	**and** $(a+b)^2 = a^2 + 2ab + b^2$ **(math rule)**
↓	
O $f(t+ \Delta t)$	**so** $(t + \Delta t)^2 = t^2 + 2t\,\Delta t + (\Delta t)^2$

So distance traveled is $5(t^2 + 2t\,\Delta t + \Delta t^2) - 5t^2$

or combine to get $5(t^2 + 2t\,\Delta t + \Delta t^2 - t^2)$

so distance to travel from t to $(t + \Delta t)$ **is** $5(2t\,\Delta t + \Delta t^2)$

ARE YOU WITH ME? CHECK OUT THE ALGEBRA AGAIN!
Now to find the velocity (where we were headed)

$$\text{velocity} = \frac{\Delta d}{\Delta t} \qquad \text{so} \quad \frac{5(2t\,\Delta t + \Delta t)^2}{\Delta t}$$

so velocity $= 5(2t + \Delta t)$ **but as** Δt **gets really small**

velocity $= 10t$ **(at least very close)**

so the speed of the ball at any time t is 10t
ie 1 sec = 10m/s, 2 sec = 20m/s .....

so $f(t) = 5t^2$ **is the distance**
and $f'(t) = 10t$ **is the velocity**

Cool huh ? The velocity, f', is the derivative of the distance! So we found the velocity at any time t from the distance formula.

Hey, it works for anything that has a *rate change!*

velocity, cooling, forces, economics, sociology engineering stock market, rainbows, etc........

Derivatives and Derivative Rules

Remember velocity is f'(t) and came from distance function f(t) (we could say v(t) is the derivative of x(t)!)

and $f'(t) = \dfrac{f(t + \Delta t) - f(t)}{\Delta t}$

now we say $f'(t) = \lim\limits_{\Delta t \to 0} \dfrac{f(t + \Delta t) - f(t)}{\Delta t}$

HEY! THIS IS THE THEORETICAL DEFINITION OF A DERIVATIVE!!

By the way, the lim means the time is tending for Δt going towards zero from above and below zero.
but, not equal to 0 but very small and close to it.

For any function f, f' represents the instantaneous *rate of change* of the function. (v(t) is same to x(t)

Derivative Rules

Name	Formula	Example
Power rule	$f(t) = t^n$ $f'(t) = n\, t^{n-1}$ n is real # $n \neq 0$	$f(t) = t^4$ $f'(t) = 4\, t^3$
Multiplier rule	$(af)' = af'$ a is a constant multiplier	$(a3t^2)' = a(3t^2)'$ $= 6at$
Sum rule	$(f + g)' = f' + g'$	$(3t^2+2t^4)' =$ $(3t^2)'+(2t^4)'$ $= 6t + 8t^3$

Remember earlier when it took several steps to get f' from f calculating $f(t) = 3t^2$ and $f'(t) = 6t$

Now we just use the multiplier rule and the power rule to compute

$$(3t^2)^1 = 3(t^2)^1 = 3 \times 2t = 6t \qquad \textbf{EASY STUFF!!}$$

Note when using the power rule t is just t. It can't be some complicated expression involving t.

Notice also that the *power rule* tells us that the derivative of a constant is ZERO remember that!! ie. $f(t) = 10$ then $f'(t) = 0$

Some Examples

Find f'(t) when $f(t) = 666t^4 - 2t^{100}$

$$f'(t) = 4\,(666)t^3 - 200t^{99} \qquad \textbf{wow!}$$

Calculus does not just have to do with things with respect to time, but can be anything that has a rate change.

We could say $y=f(x)$ **and**

$$y' = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

where x can be any quantity that changes

We can even write the formula in a simpler way

the numerator becomes Δf and the derivative becomes

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

or even more compact $f'(x) = \dfrac{df}{dx}$

if $y = f(x)$ $\quad y' = \dfrac{dy}{dx}$

How about some practice:

1. If dist $= f(t) = 3t^2 + 4t^3$ What is velocity after 100 seconds?

2. $T(d) = 160 - \frac{1}{2} d^2$ is change in temp with respect to distance

What is temp ½ inch from center?

What is rate of change of temp at that point?

3. $f(t) =$ 1/10,000,000 $(5t^2 + 3)^2$ is population change over time

How many people are there after 60 minutes?

What is rate after 60 minutes?

4. $y = 3x^2 - 3x + 2$ find y'

5. $y = 4x^3 - \frac{1}{2} x + 2$ find $\frac{dy}{dx}$ (that is y')

6. Find $\frac{dy}{dx}$ if $y = 1{,}000{,}000$

7. $y = \pi^2$ find y'

8. $y = \frac{1}{2}\pi^2 x - 1/3\pi^3$ find $\frac{dy}{dx}$

9. $y = 1/3\pi^3 x - \quad ax^3$ find y' when a=4

10. $y=1/3\pi^3 x^2 - \frac{1}{2} ax^4$ find $\frac{dy}{dx}$ when a=4

Solutions:

1. $6t + 12t^2$ 120,600 m/s
2. T at 1/2 inch is 159 7/8 degrees
   rate is -d at 1/2 inch is -1/2 degree per inch
3. 32 people after 60 minutes.
   to find the rate, lets expand $(5t^2 + 3)^2$ then take the derivative using the power rule the derivative becomes $1/10{,}000\,(100t^3 + 60t)$, so after 60 minutes we have about 2 people/minute.
4. 6x-3
5. $12x^2 - 1/2$
6. 0
7. 0, because derivative of a constant is zero (π^2 constant)
8. $1/2\,\pi^2$
9. if a = 4 then derivative is
   $\pi^3/3 \ - 3ax^2/2 \ = \ \pi^3/3 \ - 6x^2$
10. At a = 4

$$\frac{2\pi^3}{3} x - 2ax^3 = \frac{2\pi^3}{3} x - 8x^3$$

Derivatives and slopes

y= mx +b where m is the slope and b is the y-intercept

if y = f(x)   find the slope of a point on the curve (tangent)

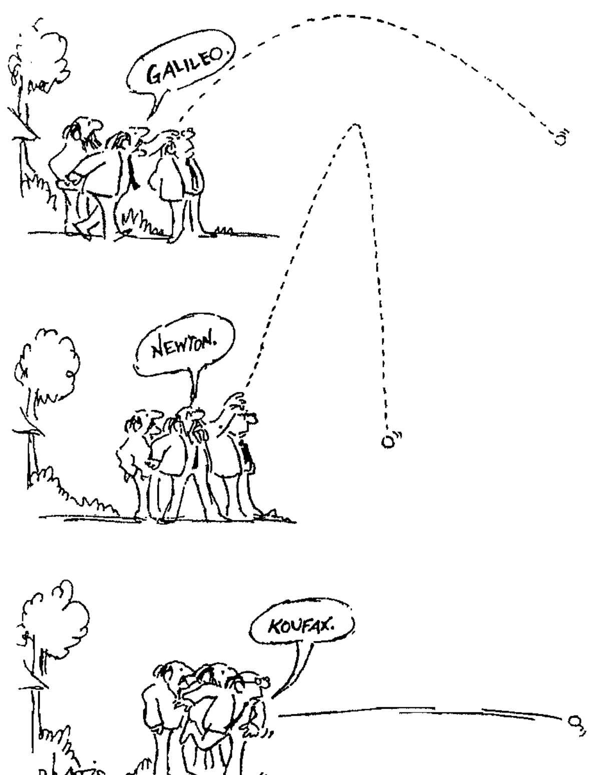
$$\frac{\Delta y}{x} = \frac{y_1 - y_0}{x_1 - x_0}$$

Now $\frac{\Delta y}{\Delta x}$ **is** $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}$

So still the slope is *the derivative of the function* ☺

if $y = f(x) = x^2$ **then** $f'(x) = 2x$

Where is the slope zero for $y = f(x) = x^2 - 4x - 6$ **?**



Rules for Products and Quotients

Suppose we asked you to do the following problem

$(3x^5 + 2x^3 - 7x)(x^4 + x^3 + 3x^2 + 1)$ find the derivative? ugh!

Stop! don't multiply the 2 big functions, use a new rule

Product and Quotient Rule Table

Rule Name	Formula
Product rule	$(fg)' = f'g + fg'$
Quotient rule	$(f/g)' = \dfrac{gf' - fg'}{g^2}$

So lets use the *product rule* to solve the problem

$(3x^5 + 2x^3 - 7x)(x^4 + x^3 + 3x^2 + 1)$
let f be the first function and g the second

$f' = 15x^4 + 6x^2 - 7$ and $g' = 4x^3 + 3x^2 + 6x$

use rule $y' = (fg)' = f'g + fg'$ and at $x = 0$ derivative is
$(-7)1 + 0$ or -7

Quotient Rule

$(3x^5 + 2x^3 - 7x)/(x^4 + x^3 + 3x^2 + 1)$

then $(f/g)' = \dfrac{gf' - fg'}{g^2}$

Now you try one $f \; \dfrac{x}{x^3 - 1}$ find derivative

How about another one $y = (x^2 + 1)(x^2 + 2)$ find derivative and slope when $x = 0$

Chain Rule and Implicit Differentiations

<u>Chain rule</u> - allows us to differentiate more complicated functions including functions of functions.

say $y = (x^3 + x + 1)^{25}$ wow, hard if we have to multiply the function inside the ( ) 25 times before finding the derivative

But, the chain rule helps, we say $y= f(u)$ and $u = g(x)$

so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ now look at y again

$y= (x^3 + x + 1)^{25}$ say $u = (x^3 + x + 1)$

so now the problem becomes

$$y = u^{25} \text{ and } dy/du = 25u^{24}$$

and $\frac{dy}{dx} = 3x^2 + 1$

use chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 25u^{24}(3x^2 + 1)$$

now sub for us and get $\frac{dy}{dx} = 25(x^3 +x +1) (3x^2 +1)$

Another way of writing <u>chain rule</u> is

$$d\ f(g(x)) = f' [g(x)]g'(x)$$

Trigonometric Derivatives

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

now <u>sine and cosine chain rules</u>

or Trig Chain Rules

$(\sin u)' = (\cos u)\, u'$
$(\cos u)' = -(\sin u)u'$

Try it yourself

find the derivative of $100(\sin x)$

find the derivative of $\cos(10x^2)$

Implicit Differentiation

Sometimes we want to find dy/dx but it is inconvenient or impossible to actually solve for y so that it looks like $y = f(x)$

ie $y^3 - xy + 7x = 15$ **hard but we can still differentiate**

say $y^3 - xy + 7x - 15 = 0$ and use chain rule. Look at y^3 part of the equation first. Its derivative is

$3y^2$ (dy/dx) where we say derivative of y is dy/dx

the derivative of y (remember the chain rule)

derivative of u^3 is $(u^3)' = 3u^2\, u'$ now apply y

$(y^3)' = 3y^2\, y'$

but what about the derivative of xy

use product rule $(fg)' = f'g + fg'$
let f be x and g be y

get $1 \cdot (y) + x \cdot dy/dx$, the last term is then $7x$

and has no y, so derivative is just 7

lets put it all together

$$3y^2 \frac{dy}{dx} - 1y - x \frac{dy}{dx} + 7 = 0$$

now solve for dy/dx = **(ans)** $\frac{y-7}{3y^2 - x}$

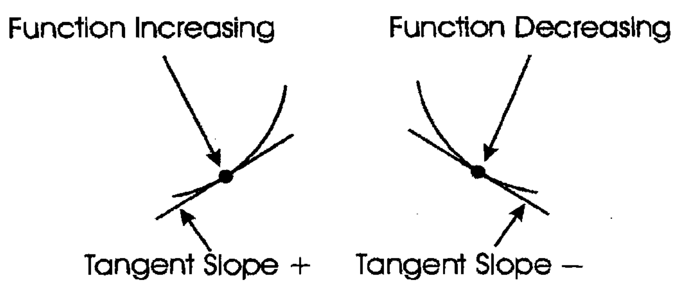
Maxima and minima and the shape of curves

imagine a curve of $y=f(x)$

if $f'(x) > 0$ at a point on curve function is increasing at point x and slope of tangent line is positive.

if $f''(x) < 0$ function decreasing at point x and slope of tangent line is negative.

Pictures of what I am saying



But if the derivative of a function "equals zero" then the slope is zero and is not increasing or decreasing. The tangent is horizontal and we call that point a *critical point.*

Think about the bottom of a parabola at the origin (0,0). The slope is decreasing until reaching (0,0) then it starts increasing

$f'(x) = 0$

So when $f'(x) = 0$, it may be a local minimum or maximum, inflection point, or just a flat curve like $f(x) = 10$

We say local minimum or maximum because the function may have a smaller minimum or a larger maximum elsewhere.

Lets try one $f(x) = x^2 + 2x - 3$ **Find all local minima**

can only occur when $f'(x) = 0$

so $f'(x) = 2x + 2$ and $= 0$ when $x = -1$

note when $x < -1$ function is decreasing $x > -1$ increasing

so when $x = -1$, $y = -4$ and it is a parabola with its minimum at $(-1,-4)$ so curve has one local min and no local maximum

Hey! there is an easier way to determine if $x = -1$ is a max or min

Just use the second derivative test. That is take the derivative of the first derivative (we call this the second derivative) denoted as f''

if $f'' > 0$ at $f(x)$ then it is a local minimum

if $f'' < 0$ at $f(x)$ then it is a local maximum

so in our problem $f'' = 2$ and is positive so it is a local minimum and graph is curved up.

Maxima and Minima tests
only valid if $f'(x_c) = 0$

Second Derivative	Type of Critcal Pont	Visual Help
$f''(x_c) > 0$	loc min at x_c	↑↓
$f''(x_c) < 0$	loc max at x_c	↓↑

Now you try one

find local minima and maxima of

$f(x) = (1/3)\, x^3 - 9x - 2$

solution: you should already have this.

$f'(x) = x^2 - 9$ and zero at $x = 3$ or -3 , so we have critical points at these two values of x.

to find out if these are maxima or minima lets take the second derivative:

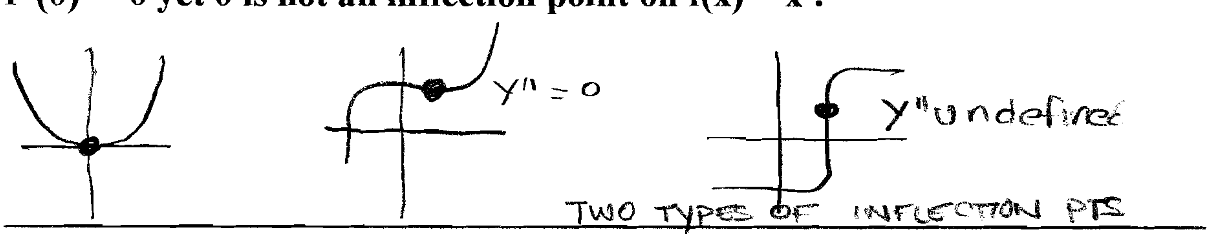
$f''(x) = 2x$ and at $x = 3$ it is positive and we have a minima at $x = -3$ it is negative and we have a maxima

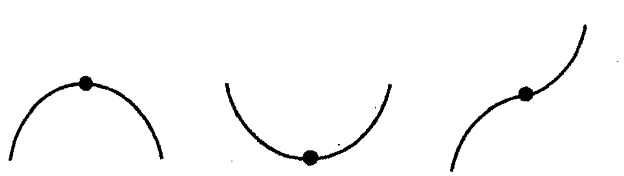
at points where $f''(x)$ is zero, we have no slope (zero) and we call that an *inflection point*

If you draw the curve, you should be able to see the max and min

But what happens when $f''(x) = 0$, it is an inflection point (switching concavity)

Plese note that Mr Perrin says this is not technally true. But that $f''(x)$ needs to change signs (from + to – or – to +) around x. For example $f(x) = x^4$ and $f''(0) = 0$ yet 0 is not an inflection point on $f(x) = x^4$.





Maximum Minimum Inflection Point

<u>Rules about inflection points</u>

1. **If f'' is an inflection pt, it must equal zero or be undefined.**
2. **f'' and f' need not exist at an inflection pt, but the function must be defined at an inflection pt.**

3. **To find inflection pts of f(x), find all pts where f''(x) = 0 and all pts where f''(x) does not exist but f(x) is defined. To check whether each of the candidates is an inflection pt, determine whether sign of f''(x) changes at those pts.**

Note **Sometimes functions like $f(x) = x^{2/3}$ have a critical pt where f' does not exist. Say the f' might be undefined because you have to divide by zero. It has a min at x=0 even if f' is not defined at x=0**

say $f'(x) = 2/3\ x^{-1/3}$ which forces us to divide by 0

$f'(0) = 2/3x^{1/3} =$

Sometimes you need to just plot the function to get a feel for the behavior

Concavity

Say we drop an ant into a bowl. The second derivative is positive everywhere the ant walks (except in dead center). Same goes for inside a parabola .

so when f'' is positive, we say surface is concave up

but we can have a function that is increasing or decreasing in slope depending on what side of the bowl or parabola you are on. It is still always concave up if f' is positive or negative

Now turn bowl over and f'' is negative so concave down and still can have f' can be positive or negative

Concavity and Slope

f'	f''	Classification	Picture
f' (x)<0	f''(x)>0	Curve decreasing concave upward	left side of U
f'(x)>0	f''(x)>0	Curve increasing concave upward	right side of U
f'(x) <0	f''(x)<0	Curve decreasing concave downward	right side of ∩
f'(x)>0	f''(x)<0	Curve increasing concave downward	left side of ∩

Lets try one

Find and describe the nature of critical points for $y = x^3$

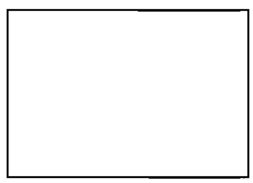
Minimum and Maxima Applications

One of the fun types of problems you can solve with calculus is maxima and minima applications for everyday things.

Lets grow some tomatoes in a garden. Say I have 50 ft of fence and I want a rectangular garden attached to my house where I can maximize the area of the garden. Assume the house is one side of the fence, how do we maximize the size of our garden?

drawing House

50-x/2



50-x/2

x

If one side is x then,
What is length of other 2 sides? 50 – x /2

Check your math (50-x/2)2 = 50

Now find a formula for the area of the rectangle a = l x w

$$a(x) = x(50-x/2) = \tfrac{1}{2}(50x - x^2)$$

But x can't be just any number we have only 50 ft of fence total

so max x is 50 and min is 0

So to maximize the area what do we do? We know maxima occours at critical pts where f'(x)=0 , so for the garden we want to maximize function describing area

Say a'(x) = ½ (50-2x) or a'(x) = 25 – x

so critical pt is at x = 25 which is our local maximum

because a"(x) = -1 negative so it is a maxima

so max area is 312.5 ft squared.

Exponents and Logarithms

Chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

remember $\log_b x$ **think of power to raise b to produce x**

ie $\log_3 9 = 2$ **because** $3^2 = 9$

or $b^y = x$ **if** $y = \log_b x$

<u>Rules for logarithms</u>

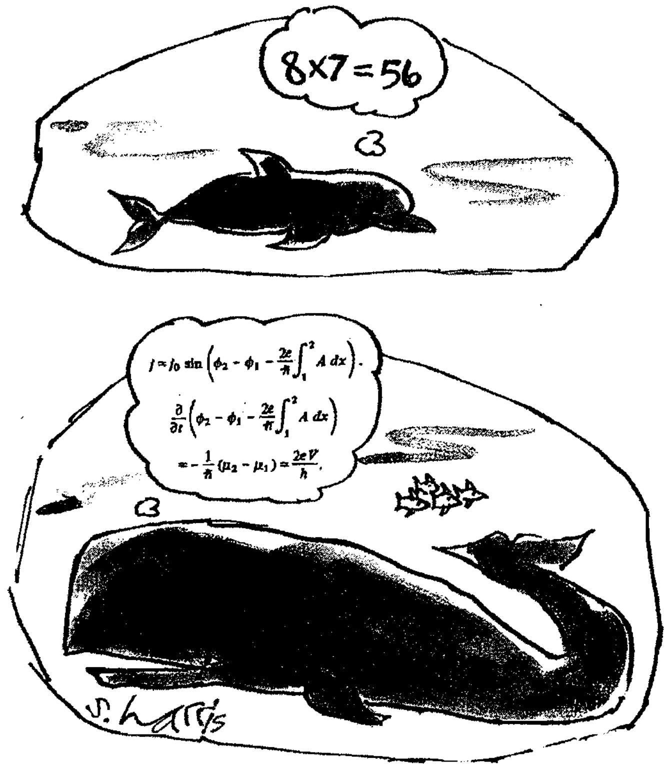
Rule Name	Logarithm Rule
Product	$\log_b xy = \log_b x + \log_b y$
Quotient	$\log_b x/y = \log_b x - \log_b y$
Reciprocal	$\log_b (1/x) = - \log_b x$
Power	$\log_b x^p = p\log_b x$
Base Change	$\log_c x = \log_b x / \log_b c$
Addition	none $\log_b (x + y)$ does not work

Natural log lnx is log base e or $\log_e x$ **(e is Eulers number 2.718…….0**
Hey, pronounce Eulers as "Oilers" for Mr Perrin

Exponential Rules

Rule Name for Derivative	Exponential Rule
Exponential	$(e^x)' = e^x$
Exponential chain	$(e^u)' = e^u u'$
Exponential base b	$(b^x)' = (\ln b)b^x$
Exponential base b chain	$(b^u)' = (\ln b)b^u u'$
Logarithm	$(\ln x)' = 1/x$
Logarithm chain	$(\ln u)' = (1/u)u'$
Logarithm base b	$(\log_b x)' = (1/\ln b)(1/x)$
Logarithm base b chain	$(\log_b u)' = (l/\ln b)(1/u)u'$

Anything else we can fake for right now. Most of the material we will learn by doing problems. Stay ahead and refer to the Quick Calculus book assigned last summer for help. Smile



Integration

Think of finding a derivative as going forward and the antiderivative as going backward.

Derivatives and antiderivatives

if v(t) = 10 t , what is f(t) or better x(t)

remember f'(t) = v(t)

so f is an antiderivative of v where f(t) = 5t^2 and we can add a constant c that does not effect rate of change. The constant may just represent not starting at zero velocity or zero displacement.

If v(t) = 10t $f(t) = \dfrac{10t^{n+1}}{n+1} = \dfrac{10t^{1+1}}{1+1} = \dfrac{10t^2}{2} = 5t^2$

Remember we call f(t) the distance or displacement and we could refer to it as x(t), which we almost always do!

Antiderivatives

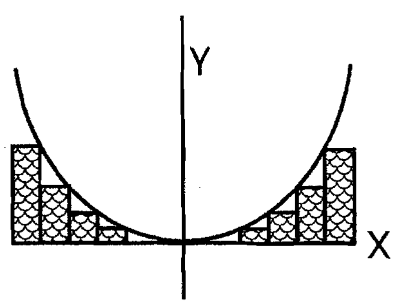
Function	**Antiderivative**
0	**C**
a	**ax + c**
x	**$\frac{1}{2}x^2 + c$**
x^r	**$x^{r+1}/r + 1 + c$ (r cannot equal -1)**
x^{-1}	**ln |x| + c (x cannot equal 0)**
cos ax	**1/a sin ax + c**
sin ax	**-1/a cos ax + c**
e^{ax}	**1/a e^{ax} + c**

Integrals

2 kinds *definite* – of a function is a specific number (area under a curve) very important!

***indefinite* – antiderivatives gives another function**

Integrals help find areas under curves defined by functions



Say $y = x^2$ find area between x = -1 and x = 1

Height of each rectangles is f(x)
Sum of all the areas is $\Sigma f(x) \Delta x$ where Δx is width of rectangle

area under a curve = $\lim_{\Delta x \to 0} \Sigma f(x) \Delta x$

We call this the Riemann integral and use the symbol $\int$

$$\int f(x)\,dx = \lim_{\Delta x \to 0} \Sigma f(x)\,\Delta x$$

from point a to point b (interval)

We can compute $\int f(x)\,dx$ for any continuous function f between a and b by just finding the antiderivative, f(x).

$$\int f(x)\,dx = F(b) - F(a) = F(x)]^b_a$$

Try this one

$y = x^2$ for $-1 \leq x \leq 1$

Special Note: over the next few weeks we will do a lot of practice problems and have many small quizzes. Study this material, practice a lot, and refer to your Calculus help book whenever you can. I will give a copy of this to the Calculus teachers so they know what direction we are headed in. Don't panic if some of the math is unfamiliar or difficult, we will work through those issues.